A Flexible Tensor Block Coordinate Ascent Scheme for Hypergraph Matching

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Motivation

Our Tensor BCA Framework

Results

Graph Matching in Computer Vision

Previous Works

Find correspondences between two sets of features / points?

- difficult to deal with rotation + scale
- Quadratic Assignment Problem

\[
\max_{x \in \mathcal{M}} \sum_{ij} \mathcal{F}_{ij} x_i x_j
\]

Graph Matching

Challenges: noise, deformation, geometric transformations

- both rotation and scale invariant
- more robust to noise
- Higher Order Assignment Problem

\[
\max_{x \in \mathcal{M}} \sum_{ijk} \mathcal{F}^3_{ijk} x_i x_j x_k
\]

Hypergraph Matching

\[ M = \text{discrete one-to-one mapping constraint} \]
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Previous Works

\[ M = \text{discrete one-to-one mapping constraint} \]

\[
\max_{x \in M} \sum_{ijk} F_{ijk} x_i x_j x_k
\]

Tensor Marginalization
(Zass and Shashua, CVPR’ 08
Chertok and Keller, PAMI’ 10)
(loss of information)

Spectral Relaxation
(Duchenne et al., CVPR’ 09)
(loose)

Continuous Relaxation
(Lee et al., CVPR’ 11)

Continuous Solution

Projection onto M
e.g. Hungarian method
(typically lead to a loss of accuracy)

Discrete One-to-one Mapping
\[ x \in M \]
Main Concepts

- An \( m \)-dimensional array \( F^m \) is called an \( m \)-order tensor
e.g. a vector is an one-dimensional tensor, a matrix is a two-dimensional tensor, etc

- One associate to any tensor \( F^m \) a multilinear form
  \[ F^m : \mathbb{R}^n \times \ldots \times \mathbb{R}^n \rightarrow \mathbb{R} \] such that
  \[
  F^m(x^1, \ldots, x^m) := \sum_{i_1 \ldots i_m} F^m_{i_1 \ldots i_m} x^1_{i_1} \ldots x^m_{i_m}
  \] (1)

  When \( x^1 = \ldots = x^m = x \), the resulting function
  \[
  F^m(x, \ldots, x) := \sum_{i_1 \ldots i_m} F^m_{i_1 \ldots i_m} x_{i_1} \ldots x_{i_m}
  \] (2)

  is called the \( m \)-order score function

Hypergraph Matching problem becomes:

\[
\max_{x \in M} \sum_{ijk} F^3_{ijk} x_i x_j x_k = \max_{x \in M} F^3(x, x, x)
\] (3)

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Tensor BCA for Hypergraph Matching
Step 1: Lifting the tensor

Given $F^3 \in \mathbb{R}^{n \times n \times n}$, the lifted 4th order tensor $F^4$ is defined as

$$F^4_{ijkl} = F^3_{ijk} + F^3_{ijl} + F^3_{ikl} + F^3_{jkl}$$  \hspace{1cm} (1)$$

$$\Rightarrow F^4(x, x, x, x) = F^3(x, x, x) \sum_{i=1}^{n} x_i$$  \hspace{1cm} (2)$$

Properties: If the new score function $F^4(x, x, x, x)$ is convex on $\mathbb{R}^n$ then

$$\max_{x,y,z,t \in M} F^4(x, y, z, t) = \max_{x,y \in M} F^4(x, x, y, y) = \max_{x \in M} F^4(x, x, x, x)$$  \hspace{1cm} (3)$$

Remarks:

1. Since $\sum_{i} x_i = \text{constant}$, $\forall x \in M$, it holds from (2)

$$\max_{x \in M} F^4(x, x, x, x) \equiv \max_{x \in M} F^3(x, x, x)$$

2. The lifting step does not cause computational overload.
Algorithm 1: \[ \max_{x,y,z,t \in M} F^4(x, y, z, t) \]

\[ F^4(x, y, z, t) \leq \max_{u \in \{x, y, z, t\}} F^4(u, u, u, u) \]

Homogenization step

1) \[ u = \arg \max_{u \in \{x, y, z, t\}} F^4(u, u, u, u) \]
2) \[ x = y = z = t = u \]

✓ Monotonic ascent for \( F^3(x, x, x) \) on \( M \)

✓ Convergence in finite number of iterations
Algorithm 1: $\max_{x,y,z,t \in M} F^4(x, y, z, t)$

\[ F^4(x, y, z, t) \leq \max_{u \in \{x, y, z, t\}} F^4(u, u, u, u) \]

**Homogenization step**

1) $u = \arg \max_{u \in \{x, y, z, t\}} F^4(u, u, u, u)$
2) $x = y = z = t = u$

- Monotonic ascent for $F^3(x, x, x)$ on $M$
- Convergence in finite number of iterations

Algorithm 2: $\max_{x, y \in M} F^4(x, x, y, y)$

\[ F^4(x, x, y, y) \leq \max_{u \in \{x, y\}} F^4(u, u, u, u) \]

**Homogenization step**

1) $u = \arg \max_{u \in \{x, y\}} F^4(u, u, u, u)$
2) $x = y = u$

- Monotonic ascent for $F^3(x, x, x)$ on $M$
- Convergence in finite number of iterations
Step 2: Tackling the convexity assumption

Consider the following multilinear form:

\[
F^4_\alpha(x, y, z, t) := F^4(x, y, z, t) + \alpha \frac{\langle x, y \rangle \langle z, t \rangle + \langle x, z \rangle \langle y, t \rangle + \langle x, t \rangle \langle y, z \rangle}{3}
\]  

(1)

Then \( F^4_\alpha(x, x, x, x) \) is convex for any \( \alpha \geq 3 \sqrt{\frac{\sum_{i,j,k,l=1}^{n} (F^4_{ijkl})^2}{n}} \). Besides, it holds for all \( \alpha \)

\[
\max_{x \in M} F^4_\alpha(x, x, x, x) \equiv \max_{x \in M} F^3(x, x, x)
\]  

(2)

To reduce the influence of \( \alpha \):

- Phase 1: run algorithm with \( \alpha = 0 \)

- Phase 2: restart algorithm with \( \alpha = 3 \sqrt{\frac{\sum_{i,j,k,l=1}^{n} (F^4_{ijkl})^2}{n}} \) with starting point initialized from output of Phase 1
Matching Quality (CMU House Dataset)

- 30 points are manually tracked over a sequence of frames of the same object taken at different view points.
- In Figures: Baseline = difficulty of the problem.

![Graph showing matching quality results]

30 points vs. 30 points

(Easy)
Matching Quality (CMU House Dataset)

- 30 points are manually tracked over a sequence of frames of the same object taken at different viewpoints.
- In Figures: Baseline = difficulty of the problem.

![30 points vs. 30 points](Easy)

![20 points vs. 30 points](Medium)

![10 points vs. 30 points](Difficult)
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Matching Quality (CMU House Dataset)

10 points vs 30 points

MPM (4/10)

TM (5/10)

RRWHM (7/10)

ours(A1) (10/10)

ours(A2) (10/10)
Motivation
Our Tensor BCA Framework
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Matching Quality (Car and Motorbike Dataset)

Motor

Car

Accuracy

Matching Score

# Outliers

HGM
TM
RRWHM
Ours (A1)
Ours (A2)

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Tensor BCA for Hypergraph Matching
Matching Quality (Car and Motorbike Dataset)

Example Input

RRWHM (28/34)

TM (10/34)

HGM (9/34)

Ours(A1) (28/34)

Ours(A2) (34/34)
Conclusion

Our Contributions

- a theoretically sound optimization framework for hypergraph matching
- two state-of-the-art algorithms with theoretical guarantees
  - monotonic ascent for original score function on $M$
  - convergence in finite number of iterations

Main Ideas

- lifting 3rd order tensor to 4th order higher order
- multilinear optimization over one-to-one constraint

Advantages

- no projection at final step
- flexible objective function

- simple yet effective

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Tensor BCA for Hypergraph Matching