Convex Optimization and Modeling	Jun.-Prof. Matthias Hein
Exercise Sheet 1 - 14.4.2010	due: 23.4.2010

Exercise 1 - Dual norm

The algebraic dual V^* of the vector space V is the set of all linear maps from V to $\mathbb{R}$. Given that the vector space $V = \mathbb{R}^n$ is equipped with a norm $\|\cdot\|$ one defines a **dual norm** on the dual space $(\mathbb{R}^n)^*$ as

$$\|v\|^* = \sup_{u \in \mathbb{R}^n} \Big\{ \sum_{i=1}^n v_i u_i \;\Big|\; \|u\| \leq 1 \Big\}.$$

This is basically the operator norm of the linear map, $v : \mathbb{R}^n \to \mathbb{R}$, discussed in the lecture.

a. **(2 Points)** Derive the dual norm of the l_1-norm, $\|u\|_1 = \sum_{i=1}^n |u_i|$.

b. **(2 Points)** Derive the dual norm of the l_2-norm, $\|u\|_2 = \sqrt{\sum_{i=1}^n u_i^2}$.

c. **(2 Points)** Derive the dual norm of the l_∞-norm, $\|u\|_\infty = \max_{i=1,\ldots,n} |u_i|$.

Hint:

- first prove a lower bound for $\|v\|^*$ by plugging in a particular u, then prove an upper bound on $\|v\|^*$ and show that upper and lower bound agree,
- for *b*) you may use the **Cauchy-Schwarz inequality**

  $$|\langle u, v \rangle| \leq \|u\|_2 \, \|v\|_2 \,,$$

  or in components

  $$|\sum_{i=1}^n u_i v_i| \leq \sqrt{\sum_{i=1}^n u_i^2} \sqrt{\sum_{i=1}^n v_i^2}.$$

Exercise 2 - Reminder of Linear Algebra and Analysis

a. **(3 Points)** Proof the assertion from the lecture that every real, symmetric matrix A has the decomposition

$$A = Q \Lambda Q^T,$$

where Q is an orthogonal matrix and Λ is a diagonal matrix having the eigenvalues on the diagonal.

b. **(3 Points)** The distance of a point x to a set C is defined as

$$d(x, C) = \inf \big\{ \|x - y\| \;\big|\; y \in C \big\}.$$

Let C be closed. Prove that the distance $d(x, C)$ is realized by an element of C that means $\exists z \in C$ such that $d(x, C) = d(x, z)$.