Convex Optimization and Modeling	Jun.-Prof. Matthias Hein
Solution of Exercise Sheet 6 - 19.05.2010	

Exercise 12 - Perturbation of the feasible set

a. **(2 Points)** Exercise 5.1d)

Solution:

a. The constraint $(x-2)(x-4) \leq u$ is equivalent to the feasible set $3-\sqrt{1+u} \leq x \leq 3+\sqrt{1+u}$ for $u \geq -1$. For $u < -1$ the problem is infeasible. For $u \geq 8$ the feasible set contains the global minimum of the objective function at $x = 0$ and thus $p^* = 1$. Thus we get

$$p^*(u) = \begin{cases} 1, & \text{for } u \geq 8, \\ 11 + u - 6\sqrt{1+u}, & \text{if } -1 \leq u \leq 8, \\ \infty & \text{if } u < 1. \end{cases}$$

The function $p^*(u)$ is differentiable for $-1 < u < 8$ and

$$\frac{\partial p^*}{\partial u} = 1 - \frac{3}{\sqrt{1+u}}, \quad \Longrightarrow \quad \frac{\partial p^*}{\partial u}\Big|_0 = -2 = -\lambda^*.$$

Exercise 13 - Barrier method

This exercise shows to construct an unconstrained problem from a constrained problem. This will be the idea of the interior point method/barrier method, which we will discuss next.

a. **(2 Points)** Exercise 5.15.

Solution:

a. The function $h_i(f_i(x))$ is convex as it is the concatenation of a non-decreasing convex function h_i and a convex function f_i. As the objective is convex and a sum of convex functions is convex, the function $\phi(x) = f_0(x) + \sum_{i=1}^{s} h_i(f_i(x))$ is convex.

If x^* is optimal for ϕ, a necessary and sufficient condition is,

$$\nabla f_0(x^*) + \sum_{i=1}^{m} h_i'(f_i(x^*))\nabla f_i(x^*) = 0,$$

where $h_i'(f_i(x^*)) \geq 0$ for all $i = 1, \ldots, s$.

The Lagrange function of the original problem is

$$L(x, \lambda) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x).$$

and as the dual Lagrange function is defined as $q(\lambda) = \inf_x L(x, \lambda)$ we obtain, that $\lambda_i' = h_i'(f_i(x^*)), \quad i = 1, \ldots, s$ is dual feasible.

The corresponding lower bound is given by,

$$q(\lambda') = f_0(x^*) + \sum_{i=1}^{m} h_i'(f_i(x^*))f_i(x^*).$$

Note, that x^* needs not to be feasible for the original primal problem. But as the h_i become more and more steep at 0, at some point x^* will become feasible for the original problem. This will be the basic building principle for the barrier method.

Exercise 14 - A problem where strong duality fails

a. **(3 Points)** Exercise 5.21a)-c).

Solution:

a. The objective is clearly convex and $\frac{x^2}{y}$ is the perspective of x^2 (p. 89, Example 3.18) and thus convex. Alternatively one computes the Hessian and observes that the determinant is zero (product of the two eigenvalues) and the trace (sum of the two eigenvalues) is positive and thus one eigenvalue is zero, the other one is positive and thus the Hessian is positive semi-definite on the whole domain.

b. The Lagrangian is
$$L((x,y),\lambda) = e^{-x} + \lambda\frac{x^2}{y}.$$
The Lagrangian is lower-bounded by zero and this lower bound can be arbitrarily well approximated by driving x and y to infinity ($y = x^3$ is sufficient that both terms converge to zero as $x \to \infty$). This holds for any $\lambda \geq 0$ and thus
$$q(\lambda) = 0.$$
The dual problem is thus trivial and the dual optimal value d^* is equal to 0. The optimal value p^* of the primal problem is 1 as the feasible set X is,
$$X = \{(x,y) \mid x = 0,\ y > 0\},$$
and thus the optimal duality gap $p^* - d^*$ is equal to 1.

c. Slater's condition does not hold as there exists no feasible point $(x,y) \in X$ where $\frac{x^2}{y} < 0$.

Exercise 15 - The weak min-max inequality

In order to appreciate the result of strong duality even more it is instructive to derive the weak min-max inequality.

a. **(2 Points)** Exercise 5.24.

Solution:

a. We have to prove that for $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $W \times Z \subseteq \operatorname{dom} f$ and W, Z non-empty

$$\sup_{z\in Z} \inf_{w\in W} f(w,z) \leq \inf_{w\in W} \sup_{z\in Z} f(w,z).$$

Note, that $f(w,z) \leq \sup_{z\in Z} f(w,z)$ for all $z \in Z$ and therefore

$$\inf_{w\in W} f(w,z) \leq \inf_{w\in W} \sup_{z\in Z} f(w,z),\ \forall\, z \in Z.$$

Since the above inequality holds with respect to all $z \in Z$ we can take the supremum on the left hand side,

$$\sup_{z\in Z} \inf_{w\in W} f(w,z) \leq \inf_{w\in W} \sup_{z\in Z} f(w,z).$$

Exercise 16 - Geometric Interpretation of KKT conditions

a. **(3 Points)** Exercise 5.31.

Solution:

a. Suppose that $\lambda_i^* = 0,\ i = 1, \ldots, m$, then the KKT condition implies $\nabla f_0(x^*) = 0$ which clearly implies the first order optimality condition. Thus the interesting case is where $\lambda_i^* > 0$ for at least one $i \in \{1, \ldots, m\}$. We define $S = \{i \mid \lambda_i^* > 0\}$. Then $f_i(x^*) = 0$ for all $i \in S$ and thus all these inequality constraints are active (these are not necessarily all active inequality constraints). Note, that for all $i \in S$,

$$\langle \nabla f_i(x^*), x - x^* \rangle = 0,$$

defines a supporting hyperplane at x^* of the feasible set X. This follows by the first-order condition,

$$f_i(x) \geq f_i(x^*) + \langle \nabla f_i(x^*), x - x^* \rangle = \langle \nabla f_i(x^*), x - x^* \rangle .$$

In particular for any feasible point $x \in X$ and $i \in S$,

$$\langle \nabla f_i(x^*), x - x^* \rangle \leq 0, \qquad \left\langle \sum_{i=1}^n \lambda_i^* \nabla f_i(x^*), x - x^* \right\rangle \leq 0,$$

and thus the KKT condition implies the first order optimality condition using $\nabla f_0(x^*) = -\sum_{i=1}^n \lambda_i^* \nabla f_i(x^*)$,

$$\langle \nabla f_0(x^*), x - x^* \rangle \geq 0,$$

for all feasible $x \in X$.