## Convex Optimization and Modeling

Solution of Exercise Sheet 5-12.05.2010

## Exercise 9 - Solution of simple QCQPs

a. (6 Points) Exercise 4.21. The introduction of the new variable $y=A^{-\frac{1}{2}} x$ simplifies all problems. Note additionally, that for a symmetric matrix $A$ the smallest eigenvalue $\lambda_{\min }(A)$ of $A$ is given as

$$
\lambda_{\min }(A)=\min \left\{\langle y, A y\rangle \mid\|y\|^{2}=1\right\}
$$

or equivalently,

$$
\lambda_{\min }(A)=\min _{y} \frac{\langle y, A y\rangle}{\|y\|^{2}}
$$

## Solution:

a. As $A \in S_{++}^{n}$ the mapping $y=A^{\frac{1}{2}} x$ is bijective and thus we can use $y$ as alternative variable. The problem rewritten in $y$ gives,

$$
\begin{array}{r}
\min \left\langle A^{-\frac{1}{2}} c, y\right\rangle \\
\text { subject to: }\langle y, y\rangle \leq 1
\end{array}
$$

We have by Cauchy-Schwarz, $\left\langle A^{-\frac{1}{2}} c, y\right\rangle \geq-\left\|A^{-\frac{1}{2}} c\right\|\|y\|$ and equality is attained for $y=$ $-\alpha A^{-\frac{1}{2}} c$ with $\alpha \geq 0$. Using $\|y\| \leq 1$, the minimum is clearly $p^{*}=-\left\|A^{-\frac{1}{2}} c\right\|$ and $y^{*}=$ $-\frac{1}{\left\|A^{-\frac{1}{2}} c\right\|} A^{-\frac{1}{2}} c$. Transforming back to the old variable $x$, we get $x^{*}=-\frac{1}{\left\|A^{-\frac{1}{2}} c\right\|} A^{-1} c$.
In the general case when $A \notin S_{+}^{n}$ we do an eigendecomposition of $A$ with $A=\sum_{r} \lambda_{r} u_{r} u_{r}^{T}$ and write $x=\sum_{r} \alpha_{r} u_{r}$ and get the equivalent problem,

$$
\begin{array}{r}
\min \sum_{r} \alpha_{r}\left\langle c, u_{r}\right\rangle \\
\text { subject to: } \sum_{r} \lambda_{r} \alpha_{r}^{2} \leq 1
\end{array}
$$

Let $S=\operatorname{span}\left\{u_{r} \mid \lambda_{r} \leq 0\right\}$. If $c$ is not orthogonal to $S$, then $p^{*}=-\infty$ as $\alpha$ can be driven to either minus or plus infinity.

- If $\lambda_{k}<0$ for some $k$ then $p^{*}=-\infty$ as $\alpha_{k}$ can be driven to either plus or minus infinity and thus any point becomes feasible.
- If $\lambda_{k}=0$ and $\left\langle c, u_{k}\right\rangle \neq 0$, then again $p^{*}=-\infty$,
- If $\left\langle c, u_{k}\right\rangle=0$ for all $k$ with $\lambda_{k}=0$, then the problem reduces to the one above.
b. We do again a variable transformation, $y=A^{\frac{1}{2}}\left(x-x_{0}\right)$, which is again bijective. Note that $x=x_{0}+A^{-\frac{1}{2}} y$. The equivalent problem is,

$$
\begin{aligned}
& \min \left\langle c, x_{0}\right\rangle+\left\langle c, A^{\frac{1}{2}} y\right\rangle \\
& \text { subject to: }\langle y, y\rangle \leq 1
\end{aligned}
$$

The part $\left\langle c, x_{0}\right\rangle$ in the objective is just a constant and thus the minimizer is the same as in a), $y^{*}=-\frac{1}{\left\|A^{-\frac{1}{2}} c\right\|} A^{-\frac{1}{2}} c$ and the optimal value $p^{*}=\left\langle c, x_{0}\right\rangle-\left\|A^{-\frac{1}{2}} c\right\|$. Transforming into the original variable we get

$$
x^{*}=x_{0}-\frac{1}{\left\|A^{-\frac{1}{2}} c\right\|} A^{-1} c
$$

c. Using again that $A \in S_{++}^{n}$ we do the variable transformation $y=A^{\frac{1}{2}} x$ and get the problem,

$$
\begin{aligned}
& \min \left\langle y, A^{-\frac{1}{2}} B A^{-\frac{1}{2}} y\right\rangle \\
& \text { subject to: }\langle y, y\rangle \leq 1
\end{aligned}
$$

Now, we know that $\lambda_{\min }(A)=\min \left\{\langle x, A x\rangle \mid\|x\|^{2}=1\right\}$. Thus if the minimum is attained at the boundary then it is the minimal eigenvalue of $A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ which is non-negative as $B \in S_{+}^{n}$ and $A \in S_{++}^{n}$. If the minimum is attained inside, the gradient of the objective,

$$
\nabla f(y)=2 A^{-\frac{1}{2}} B A^{-\frac{1}{2}} y
$$

has to vanish which is the case for any element of the null space of $A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ in particular $y=0$. Thus $p^{*}=0$ as the smallest eigenvalue is non-negative and one solution is always $y^{*}=0$ and thus $x^{*}=0$.
In the case where $B \notin S_{+}^{n}$ we get,

$$
p^{*}= \begin{cases}\lambda_{\min }\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right), & \text { if } \lambda_{\min }\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) \leq 0 \\ 0, & \text { else. }\end{cases}
$$

## Exercise 10 - Robust quadratic programming

Robust programming is useful in many applications where full knowledge of the problem parameters is unavailable. Robust programming optimizes the worst case over all possible parameter possibilities.
a. (4 Points) Exercise 4.28a) and b). For b) it might be helpful to first consider the effect of such a perturbation on a fixed vector $x$,

$$
\left\langle x,\left(P-P_{0}\right) x\right\rangle .
$$

## Solution:

a. $E=\left\{P_{1}, \ldots, P_{K}\right\}$, where $P_{i} \in S_{+}^{n}, i=1, \ldots, K$. The robust quadratic program can be formulated as

$$
\begin{aligned}
& \min t \\
& \text { subject to: } \frac{1}{2}\left\langle x, P_{i} x\right\rangle+\langle q, x\rangle+r \leq t, \quad i=1, \ldots, K \\
& A x \preceq b .
\end{aligned}
$$

Since we have a quadratic inequality constraint and a linear objective, we have a QCQP.
b. $E=\left\{P \in S^{n} \mid-\gamma \mathbb{1}_{n} \preceq P-P_{0} \preceq \gamma \mathbb{1}_{n}\right\}$. We have

$$
-\gamma \mathbb{1}_{n} \preceq P-P_{0} \preceq \gamma \mathbb{1}_{n} \quad \Longleftrightarrow \quad-\gamma\|x\|^{2} \leq\left\langle x,\left(P-P_{0}\right) x\right\rangle \leq \gamma\|x\|^{2}, \quad \forall x \in \mathbb{R}^{n} .
$$

Thus, we get,

$$
\begin{equation*}
\sup _{P \in E}\langle x, P x\rangle=\left\langle x, P_{0} x\right\rangle+\sup _{P \in E}\left\langle x,\left(P-P_{0}\right) x\right\rangle \leq\left\langle x, P_{0} x\right\rangle+\gamma\|x\|^{2} \tag{1}
\end{equation*}
$$

The last inequality is actually an equality since for an arbitrary vector $e_{z}$ with $\left\|e_{z}\right\|=1$ we have $P=\gamma e_{z} e_{z}^{T}+P_{0} \in E$ since $e_{z} e_{z}^{T} \preceq \mathbb{1}_{n}$ which is equivalent to,

$$
\left\langle x,\left(P-P_{0}\right) x\right\rangle=\gamma\left\langle x, e_{z}\right\rangle^{2} \leq \gamma\|x\|^{2}\left\|e_{z}\right\|^{2}, \quad \forall x \in \mathbb{R}^{n}
$$

and we have equality if, $x=\lambda e_{z}$, with $\lambda \geq 0$. Thus, for any fixed $x$ we can find a $P$ such that equality in Equation (1) holds. In total, we have

$$
\sup _{P \in E}\langle x, P x\rangle=\left\langle x,\left(P_{0}+\gamma \mathbb{1}_{n}\right) x\right\rangle .
$$

Thus we get the problem,

$$
\min \frac{1}{2}\left\langle x,\left(P_{0}+\gamma \mathbb{1}_{n}\right) x\right\rangle+\langle q, x\rangle+r
$$

subject to: $A x \preceq b$,
which is a QP.

## Exercise 11 - Lagrangian, dual problem and strong duality

a. (5 Points) Exercise 5.1a)-c).

## Solution:

a. - the domain $D$ of the objective function is $D=\mathbb{R}$,

- We have the inequality $(x-2)(x-4) \leq 0$ and thus the feasible set is $X=\{x \mid 2 \leq x \leq 4\}$,
- The quadratic objective is strictly increasing on [2,4] thus the optimal solution is $x^{*}=2$ and the optimal value $p^{*}=5$.
b. The Lagrangian is given as

$$
L(x, \lambda)=x^{2}+1+\lambda\left(x^{2}-6 x+8\right)=(1+\lambda) x^{2}-6 \lambda x+(8 \lambda+1)
$$

where $\lambda \in \mathbb{R}_{+}$. The Lagrangian is a convex function in $x$ with positive sign for $\lambda>-1$ and thus we can easily determine the minimum in $x$ as the stationary point of the Lagrangian,

$$
\frac{\partial L}{\partial x}(x, \lambda)=2(1+\lambda) x-6 \lambda=0
$$

This leads to $x=\frac{3 \lambda}{1+\lambda} \in D$. For $\lambda \leq-1$ the Lagrangian is unbounded from below and thus the dual function is given as

$$
q(\lambda)=\inf _{x \in D} L(x, \lambda)= \begin{cases}\frac{-\lambda^{2}+9 \lambda+1}{1+\lambda} & \text { for } \lambda>-1 \\ -\infty & \text { else }\end{cases}
$$

We verify that: $q(\lambda) \leq p^{*}$. We have

$$
\frac{-\lambda^{2}+9 \lambda+1}{1+\lambda} \leq 5 \quad \Leftrightarrow \quad 5(1+\lambda) \geq-\lambda^{2}+9 \lambda+1 \quad \Leftrightarrow \quad(\lambda-2)^{2} \geq 0
$$

and since this holds for all $\lambda$ we have weak duality, $\forall \lambda \geq 0, \quad q(\lambda)=d^{*} \leq p^{*}$.


Figure 1: Left: the constrained problem, Right: the dual function.

