Convex Optimization and Modeling — Jun.-Prof. Matthias Hein

Solution Exercise Sheet 2 - 21.4.2010

Exercise 3 - Convex sets

a. **(7 Points)** Solve 2.12 in BV.

Solution:

- The set $\{x \,|\, \alpha \leq \langle x, w\rangle \leq \beta\}$ for $w \in \mathbb{R}^n$ is the intersection of two half-spaces and therefore convex,
- A rectangle is a polyhedron or alternatively the intersection of four half-spaces,
- Intersection of two half spaces,
- For one point the set
$$\big\{x \,|\, \|x - x_0\|_2 \leq \|x - y\|_2\big\},$$
is convex
$$\|x - x_0\|_2 \leq \|x - y\|_2 \quad \Leftrightarrow \quad \langle x, x_0 - y\rangle \leq \frac{1}{2}\Big(\|y\|^2 - \|x_0\|^2\Big),$$
since it is a linear half-space. One can rewrite
$$\big\{x \,|\, \|x - x_0\|_2 \leq \|x - y\|_2, \forall\, y \in S\big\} = \bigcap_{y \in S} \big\{x \,|\, \|x - x_0\|_2 \leq \|x - y\|_2\big\}.$$
Since the intersection of convex sets is convex we are done.
- This set is generally not convex. A simple counterexample is the following: take S as a half-circle (centered at the origin, the "left part" of the circle) and T as the line starting at the origin. Then the ends of the half-circle are clearly in the set but the line connecting them touches the origin which is element of T.
- Let $M = \big\{x \,|\, x + S_2 \subset S_1\big\}$ where S_1 is convex and S_2 arbitrary. Let $x, y \in M$, then we have
$$x + w \in S_1,\ \forall\, w \in S_2 \text{ and } y + w \in S_1,\ \forall\, w \in S_2,$$
Then we have
$$\lambda x + (1 - \lambda)y + w = \lambda(x + w) + (1 - \lambda)(y + w).$$
By the first assertion we know that $x + w$ and $y + w$ are in S_1 for all $w \in S_2$. But S_1 is convex and therefore a convex combination of points is again in S_1. Thus M is convex.
- The set $\big\{x \,|\, \|x - a\| \leq \theta\, \|x - b\|\big\}$ can be rewritten as
$$\|x - a\|^2 \leq \theta^2 \|x - b\|^2 \quad \Longleftrightarrow \quad \|x\|^2 - 2\langle x, a\rangle + \|a\|^2 \leq \theta^2 \|x\|^2 - 2\theta^2 \langle x, b\rangle + \theta^2 \|b\|^2,$$
which leads to the quadratic inequality
$$(1 - \theta^2)\|x\|^2 - 2\big\langle x, a - \theta^2 b\big\rangle + \|a\|^2 - \theta^2 \|b\|^2 \leq 0.$$
Now we do a quadratic extension,
$$\left\|\sqrt{1 - \theta^2}\, x - \frac{a - \theta^2 b}{\sqrt{1 - \theta^2}}\right\|^2 - \frac{\|a - \theta^2 b\|^2}{1 - \theta^2} + \|a\|^2 - \theta^2 \|b\|^2 \leq 0.$$
The division by $1 - \theta^2$ which is positive by assumption yields,
$$\left\|x - \frac{a - \theta^2 b}{1 - \theta^2}\right\|^2 \leq \frac{\|a - \theta^2 b\|^2}{(1 - \theta^2)^2} + \frac{\theta^2 \|b\|^2 - \|a\|^2}{1 - \theta^2}.$$
This defines a ball in $\mathbb{R}^n$ which is clearly convex. Note, that the ball is non-empty since the set contains at least a.

Exercise 4 - Supporting Hyperplane

(4 Points) Exercise 2.24 in BV.

Solution:

a. The set $C = \{x \in \mathbb{R}^2_+ \mid x_1\, x_2 \geq 1\}$ is closed and convex. A closed convex set can be equally written as the intersection of all half-spaces which contain it. The half-spaces are generated by the supporting hyper-planes of all boundary points. The normal vector at a boundary point $(x_1, \frac{1}{x_1})$ (note that at a boundary point $x_1\, x_2 = 1$) is given as $(\frac{1}{x_1^2}, 1)$ and the supporting hyperplane at this point is ,

$$\big\{(y_1, y_2) \in \mathbb{R}^2 \mid \frac{1}{x_1^2} y_1 + y_2 - \frac{2}{x_1} = 0\big\}.$$

Thus we can write C as,

$$C = \bigcap_{x_1 > 0} \big\{(y_1, y_2) \in \mathbb{R}^2 \mid \frac{1}{x_1^2} y_1 + y_2 - \frac{2}{x_1} \geq 0\big\}.$$

b. The supporting hyperplane has normal vector n at point x with $\|x\| = \infty$,

$$n_i = \begin{cases} < 0, & \text{if } x_i = 1, \\ 0, & \text{if } -1 < x_i < 1, \\ > 0, & \text{if } x_i = -1. \end{cases}$$

Exercise 5 - Polyhedral Approximation

(2 Points) Exercise 2.25 in BV.

Solution:

a. The boundary points $x_1, \ldots, x_k$ are in C as C is closed. We have $P_{\text{inner}} \subseteq C$ since the convex hull of of any elements of C is always contained in C.

b. P_{outer} is an intersection of half-spaces where each half-space contains C and thus also the intersection.