Power Method for the maximal $\ell^{p_1,\dots,p_m}$-singular value of non-negative tensor

Tensor projective norm

Given a tensor $f \in \mathbb{R}^{d_1\times d_2\times\ldots\times d_m}$ of order m and $1 < p_1,\dots,p_m < \infty$, we propose in our paper, a power method type algorithm for computing a maximizer of the following problem:

$$\boxed{\|f\|_{p_1,\dots,p_m} \coloneqq \max_{\mathbf{x}_1,\dots,\mathbf{x}_m\neq 0} \frac{|f(\mathbf{x}_1,\dots,\mathbf{x}_m)|}{\|\mathbf{x}_1\|_{p_1}\cdot\ldots\cdot\|\mathbf{x}_m\|_{p_m}}}$$

where $\|\mathbf{z}\|_{p_i} \coloneqq \Big(\sum_{k=1}^{n_i}|z_k|^{p_i}\Big)^{1/p_i}$ denotes the usual p_i-norm on $\mathbb{R}^{n_i}$ and $f(\mathbf{x}_1,\dots,\mathbf{x}_m)$ is the multi-linear form associated to the tensor f, i.e.

$$f(\mathbf{x}_1,\dots,\mathbf{x}_m) \coloneqq \sum_{j_1\in[d_1],\dots,j_m\in[d_m]} f_{j_1,\dots,j_m} x_{1,j_1}\cdot\ldots\cdot x_{m,j_m} \qquad \forall(\mathbf{x}_1,\dots,\mathbf{x}_m)\in\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}\times\ldots\times\mathbb{R}^{d_m}$$

with $[d_i] \coloneqq \{1,\dots,d_i\}$.

Matrix case: When the tensor is of order $m=2$, i.e. $f = A \in \mathbb{R}^{d_1\times d_2}$ is a matrix, the maximization problem above reduces to the so-called matrix p,q-norm, namely

$$\boxed{\|A\|_{p,q} \coloneqq \max_{\mathbf{y},\mathbf{z}\neq 0}\frac{|\mathbf{y}^t A\mathbf{z}|}{\|\mathbf{y}\|_p\|\mathbf{z}\|_q} = \max_{\mathbf{z}\neq 0}\frac{\|A\mathbf{z}\|_{p'}}{\|\mathbf{z}\|_q} = \max_{\mathbf{y}\neq 0}\frac{\|A^t\mathbf{y}\|_{q'}}{\|\mathbf{y}\|_p}}$$

with $1 < p,q < \infty$.

Remark. Provided that the convergence assumptions are fulfilled, the algorithm produces two monotonic sequences $(\lambda_-^k),(\lambda_+^k)\subset\mathbb{R}$ such that

$$\lambda_-^k \leq \lambda_-^{k+1} \leq \|f\|_{p_1,\dots,p_m} \leq \lambda_+^{k+1} \leq \lambda_+^k \qquad \forall k=1,2,\dots$$

and thus can also be used to simply obtain an upper bound on $\|f\|_{p_1,\dots,p_m}$. We use these sequences as stopping criteria, i.e. the algorithm runs until $\frac{\lambda_+^k-\lambda_-^k}{\lambda_+^k} <$ tol. For an explicit expression of λ_-^k and λ_+^k we refer to the paper cited below.

Convergence guarantee

The method is guaranteed to converge to a global maximizer of the optimization problem above if:

a) f is non-negative, i.e. $f_{j_1,\dots,j_m}\geq 0$ for all $j_1\in[d_1],\dots,j_m\in[d_m]$.

b) $1 < p_1,\dots,p_m < \infty$ and there exists $i\in[m]$ such that

$$m-1 \leq (p_i'-1)\Big(\min_{k\in[m]\setminus\{i\}} p_k - (m-1)\Big).$$

Example:

- $p_1 = 1 + t$, $p_2 = \ldots = p_m \geq (m-1)(1 + 1/t)$, $t > 0$.
- $p_1 = \ldots = p_{m-1} = (m-1) + s$, $p_m \geq \frac{m-1+s}{s}$, $s > 0$.

c) The tensor f is weakly irreducible. That is, the undirected m-partite graph $G(T) = (V, E)$ is connected where

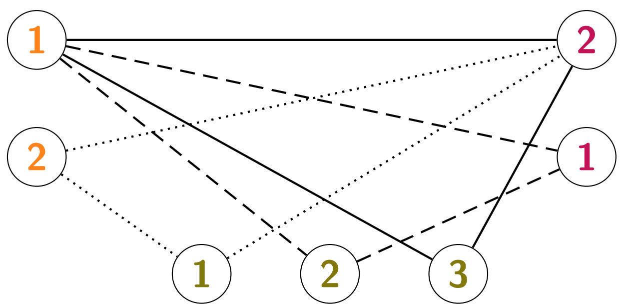
$$V \coloneqq [d_1] \sqcup [d_2] \sqcup \ldots \sqcup [d_m],$$

is the disjoint union of $[d_1], [d_2], \ldots, [d_m]$ and

$$(j_k, j_l) \in E \qquad \Longleftrightarrow \qquad \forall i \in [d] \setminus \{k, l\}, \quad \exists j_i \in [d_i] \quad \text{such that} \quad f_{j_1,\ldots,j_d} > 0.$$

Example:

$$f \in \mathbb{R}^{2\times 3\times 2} \qquad \text{with} \qquad f_{1,2,1} > 0 \qquad f_{2,1,2} > 0 \qquad \underline{f_{1,3,2}} > 0 \qquad \text{and} \qquad f_{i,j,k} = 0 \ \text{ else.}$$

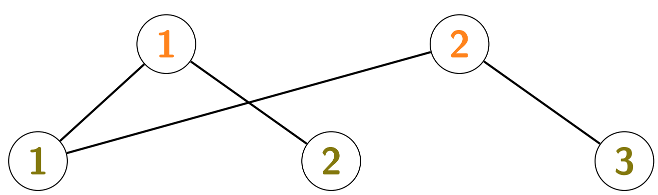


Matrix case: When the tensor $f = A \in \mathbb{R}^{d_1 \times d_2}$ is a matrix, then the three conditions above reduce to:

a) A is non-negative, i.e. $A_{i,j} \geq 0$ for every $i \in [d_1], j \in [d_2]$.

b) $1 < p, q < \infty$ and $(p-1)(q-1) \geq 1$.

c) A is weakly irreducible (see definition above).
Example: If $A = \begin{pmatrix} 1 & 2 & 0 \\ \pi & 0 & 4 \end{pmatrix} \in \mathbb{R}^{2\times 3}$, then the undirected graph $G(A)$ is given by



and therefore A is weakly irreducible because $G(A)$ is connected.

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Please include a reference to our preprint

- A. Gautier and M. Hein
  Tensor norm and maximal singular vectors of non-negative tensors - a Perron-Frobenius theorem, a Collatz-Wielandt characterization and a generalized power method
  arXiv:1503.01273

if you find this code useful for your research.